Minimal Model Program Learning Seminar.

Week 3:

- singularities of the MMP,
- Kodaira vanishing and generalizations.

02/12/2021.

MMP learning seminar, Week 3:

Singularities:

(X,Δ) log pair, X normal qp, $K_X+\Delta$ $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor

$Y \xrightarrow{\pi} X$ proj birational, Y normal qp, $E \subseteq Y$ prime

log discrepancy of (X,Δ) at E to be

$$a_E(X,\Delta) = 1 + \operatorname{coeff}_E(K_Y - \pi^*(K_X+\Delta))$$

Definition: We say that (X,Δ) is

terminal $\iff$ $a_E(X,\Delta) > 1$ for every E exc over Y.

canonical $\iff$ $a_E(X,\Delta) \geq 1$ for every E exc over Y.

Kawamata log terminal $\iff$ $a_E(X,\Delta) > 0$ for every E.

log canonical $\iff$ $a_E(X,\Delta) \geq 0$ for every E.

$\{$ Is enough to check the E's appearing on a log resolution of (X,Δ). $\}$

$$
\begin{tikzcd}
X \arrow[r, dashed, "\text{termination of flips}"] & X_{ter} \arrow[d] \\
 & X_{can}
\end{tikzcd}
$$

smooth proj

K_X pseff

$K_{X_{ter}}$ is nef.

$\Downarrow$ → abundance

$K_{X_{ter}}$ is semiample

$K_{X_{can}}$ is ample

$$X_{can} \simeq \mathrm{Proj}\left(\bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(mK_X))\right).$$

terminal sing are those that may appear in the terminal model.
canonical " " " " canonical.

Adjunction: (X,D) log smooth, $K_X+D|_D \sim K_D$.

(X,D) is log canonical but not klt.

$a_D(X,D)=0$. (purely log terminal pair).

However, $a_E(X,D)>0$ for every $E\neq D$.

- Terminal sing is the smallest category of sing that we need to understand in order to run an MMP.
- log canonical is the largest class of sing in which we expect the MMP to work.

Examples of klt sing $\longrightarrow$ Cone sing.
$\longrightarrow$ Quotient sing.

Prop: (X,Δ) is a log pair and A an ample Carlier on X.

$$C(X,\Delta) = \operatorname{Spec}\Big(\bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(mA))\Big).$$

$C(X,A)$ is terminal iff $rA \sim_{\mathbb{Q}} K_X+\Delta$ with $r<-1$ and (X,Δ) terminal

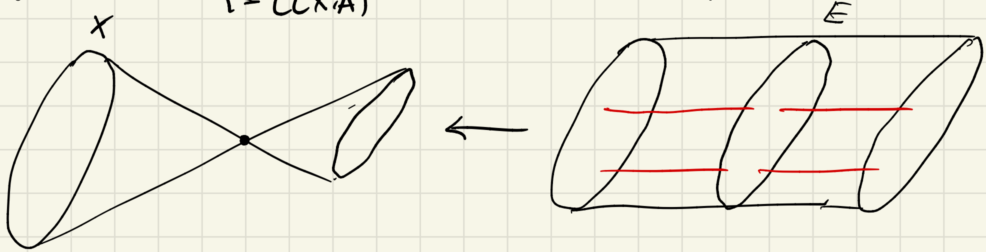
canonical : $r\leq -1$ and (X,Δ) canonical

klt $r<0$ and (X,Δ) klt

lc $r\leq 0$ and (X,Δ) is lc

Sketch:

$Y = C(X,A)$ $\qquad$ Y'



$$\pi^* K_Y = K_{Y'} + \alpha E \qquad \alpha = -\frac{1}{r}$$

$\alpha \leq 1 \iff$ lc

$\alpha < 1 \iff$ klt

$\alpha \leq 0 \iff$ canonical

$\alpha < 0 \iff$ term.

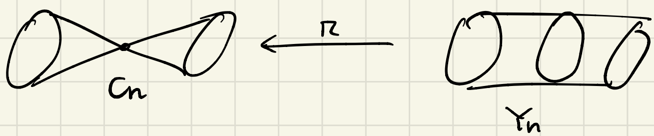
Cone over Fano is klt, Cone over CY is lc

Cone over canonically polarized is a wild sing.

Cone: Cone over norm rat curve.

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^n$$
$$[s:t] \longmapsto [s^n : s^{n-1}t : \dots : t^n]$$

Cone over the normal rat curve of degree n



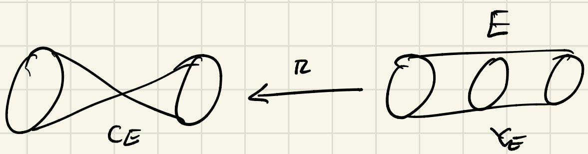
$\leftarrow$ log smooth.
(Y_n, E_n) are smooth

$E_n \simeq \mathbb{P}^1.$

$$\pi^*(K_{C_n}) = K_{Y_n} + \left(1 - \frac{2}{n}\right)E_n.$$

$$a_{E_n}(C_n) = \frac{2}{n}.$$

$E \hookrightarrow \mathbb{P}^3,$



$$\pi^*(K_{C_E}) = K_{Y_E} + E.$$

$$a_E(C_E) = 0.$$

Quotients: $G \leq GL_n(\mathbb{K})$ a finite group.

$\mathbb{C}^n/G = \operatorname{Spec}(\mathbb{K}[x_1, \dots, x_n]^G)$ has klt sing.

klt sing are preserved under finite quotients.

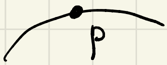
dim 1:

non-normal curve is not lc

C

normal $\Longrightarrow$ smooth $\Longrightarrow$ terminal

$(C, \alpha p)$



is klt if $\alpha < 1$

lc if $\alpha \leq 1$

$\boxed{\text{dim 2: terminal } \Longleftrightarrow \text{ smooth}}$

$x^2 + y^2 + z^n = 0.$

canonical $\Longleftrightarrow$ DuVal A_n, D_n, E_6, E_7, E_8.

klt $\Longleftrightarrow$ quotient sing $\mathbb{C}^2/G$

finite group

lc "$\Longleftrightarrow$" quotient and/or elliptic cone

hyperquotient sing.

dim 3:

terminal sing are classified: Quotients of hypersurface sing.

$$G \curvearrowright \underbrace{\{x^2 + y^2 + f(z,w) = 0\}}_{H} \qquad X = H/G.$$

analytic emb dim 4

canonical (???)

dim 4:

There are examples of 4-fold terminal sing with analytic embedding dimension n (for every n) (Kollár, 2010)

Theorem (Prokhorov, Xu, 2019): Any klt singularity deforms to a klt cone singularity.

$x \in X$, flat morphism $\mathcal{X} \xrightarrow{\varphi} \mathbb{A}^1$

$$\varphi(\mathbb{A}^1 - \{0\}) \simeq (\mathbb{A}^1 - \{0\}) \times X$$

and $\varphi^{-1}(\{0\}) = X_0$ is a klt cone sing.

this a deformation to the normal cone.
deformation to leading terms.

Philosophy: Any theorem for smooth proj var should work with klt sing.

Example: • $K_X \sim_{\mathbb{Q}} 0$, Bogomolov-Beauville. 70's

$$X \leftarrow Y \qquad Y \simeq X_1 \times \dots \times X_k.$$

Ab, irr CY, HK.

for klt sig this was proved by Druel, Campana, ... (2020).

• X smooth proj, $-K_X$ is nef 2022

$$\tilde{X} \simeq \mathbb{C}^q \times \prod Y_i \times \prod S_k \times Z$$

irr CY, HK, rationally connected.

X a variety with terminal sing and

$Z \subseteq X$ a subvariety of codim 2,

$\operatorname{Spec} \mathcal{O}_{X,Z}$ has terminal sing $\iff$ $\operatorname{Spec} \mathcal{O}_{X,Z}$ smooth local ring

dim 2 sing.

$\Updownarrow$

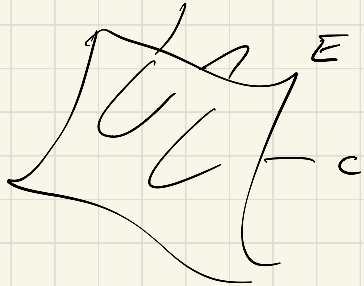
X is smooth at the generic point of Z.

If X is terminal, the sing appear in codimension ≥ 3.

$x^3 + y^2 + z^5 = 0$ this is klt codimension 2-sing.

sections help to control K_X-neg curves

K_X -negative curves. $\quad C \subseteq X$.



$K_X \sim_{\mathbb{Q}} E \geq 0$.

$K_X . C = E . C < 0$

$E . C < 0 \implies C \subseteq \mathrm{support}(E)$

$K_X + D|_D \sim K_D$. $\quad (X, D)$ is a log smooth pair

$$0 \longrightarrow \mathcal{O}_X(K_X) \xrightarrow{\otimes D} \mathcal{O}_X(K_X + D) \longrightarrow \mathcal{O}_D(K_D) \longrightarrow 0.$$

$$H^1(X, \mathcal{O}_X(K_X)) = 0 \implies \begin{array}{ccc} H^0(K_X + D) & \twoheadrightarrow & H^0(K_D) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ s_X & \longmapsto & s \end{array}$$

Vanishing thms help to produce sections.

Theorem (Kodaira vanishing): X smooth proj, $\mathcal{L}$ ample line bundle on X. Then $H^i(X, \mathcal{L}^{-1}) = 0$ $i < \dim X$.

Idea of the proof: $s \in H^0(X, \mathcal{L}^m)$ general, $D = (s=0)$ smooth.

Index one cover of $\mathcal{L}^m$ on $X \setminus D$.

$$\mathcal{O}_X \xrightarrow{s} \mathcal{L}^m \quad , \quad \bigoplus_{i=0}^{m-1} \mathcal{L}^{-i}$$

$$\mathcal{L}^{-i} \otimes \mathcal{L}^{-j} \simeq \mathcal{L}^{-i-j} \otimes \mathcal{O}_X \xrightarrow{id \otimes s} \mathcal{L}^{-i-j} \otimes \mathcal{L}^m = \mathcal{L}^{-i-j+m}.$$

$Z = \mathrm{Spec}_X \bigoplus_{i=0}^{m-1} \mathcal{L}^{-i}$, projection $p: Z \to X$.

X and D smooth $\Longrightarrow$ Z is smooth

$$\tau: H^i(Z, \mathbb{C}_Z) \twoheadrightarrow H^i(Z, \mathcal{O}_Z)$$

$$p_* \tau: H^i(X, p_* \mathbb{C}_Z) \twoheadrightarrow H^i(X, p_* \mathcal{O}_Z)$$

$$\mathrel{\rotatebox{90}{$\simeq$}} \qquad\qquad\qquad \mathrel{\rotatebox{90}{$\simeq$}}$$

$$\bigoplus_{r=0}^{m-1} H^i(X, \mathbb{C}[\xi^r]) \twoheadrightarrow \bigoplus_{r=0}^{m-1} H^i(X, \bigoplus_{r=0}^{m-1} \mathcal{L}^{-r}).$$

local systems with monodromy r

$\mathbb{C}[\xi^r] \hookrightarrow \mathcal{L}^{-r}$ factors through

$$\mathbb{C}[\xi^r] \hookrightarrow \mathcal{L}^{-r}(-kD) \hookrightarrow \mathcal{L}^{-r}.$$

$$H^i(X, \mathbb{C}[\xi^r]) \twoheadrightarrow H^i(X, \mathcal{L}^{-r}(-kD)) \twoheadrightarrow H^i(X, \mathcal{L}^{-r})$$

$$\text{Is}$$

Serre v2n $\qquad H^i(X, \mathcal{L}^{-(r+mk)}) = 0$ $\quad$ very ample

$$D \in H^0(\mathcal{L}^m)$$

holds for arbitrary k. $\square$

$$\boxed{0 = H^i(X, \mathcal{L}^{-1})} \quad i < \dim X$$

$$H^i(X, K_X \otimes \mathcal{L}) = 0 \quad \text{for } i > 0$$

$$\mathcal{L} = D.$$

Theorem (KV vanishing): X smooth proj complex.

$\mathcal{L}$ line bundle on X. $\quad \mathcal{L} \equiv \underset{\text{ample}}{M} + \underbrace{\sum a_i D_i}_{= 0}$, where

i) M big and nef $\mathbb{Q}$-divisor,

ii) $\sum D_i$ a snc divisor,

iii) $0 \le a_i < 1$, $a_i \in \mathbb{Q}$

Then $H^i(X, \mathcal{L}^{-1}) = 0$ for $i < \dim(X)$.

Proposition: X qp normal, D Cartier, m a pos natural.

$Y \xrightarrow{g} X$ finite, D' Cartier such that $g^* D \sim m D'$.

If X smooth and $\sum F_j$ is snc, then Y smooth

and $\sum g^* F_j$ snc.

Lemma. $Y \longrightarrow X$ finite, $\mathcal{O}_X \longrightarrow f_* \mathcal{O}_Y$ splits

If $\mathcal{F}$ coherent on X, then $\mathcal{F}$ is a direct summand of $f_* f^* \mathcal{F}$.

Hence, $H^i(X, \mathcal{F})$ is a direct summand $H^i(Y, g^* \mathcal{F})$.

$$\underset{0}{\parallel} \qquad \Longleftarrow \qquad \underset{0}{\parallel}$$

Sketch: $\sum_i a_i D_i$, $a_1 = b/m$, $m \geq 0$.

$p_1 : X_1 \longrightarrow X$, $p_1^* D_1 \sim mD$,

$H^i(X, \mathcal{L}^{-1})$ is a direct summand of $H^i(X_1, p_1^* \mathcal{L}^{-1})$.

$p_1^* D_1$ is a section of $\mathcal{O}_X(mD)$, so we can apply index one cover to obtain $X_2 \xrightarrow{p_2} X_1$, X_2 smooth

$p_1^*(D_i)$ smooth, $\sum_i p_2^* p_1^* D_i$ is snc.

$$p_{2*} \mathcal{O}_{X_2} = \bigoplus_{j=0}^{m-1} \mathcal{O}_{X_1}(-jD), \text{ thus}$$

$$H^i(X_2, p_2^* p_1^* \mathcal{L}^{-1}(bD)) = \bigoplus_{j=0}^{m-1} H^i(X_1, p_1^*((b-j)D)).$$

Take $j = b$, we get that

$H^i(X_1, p_1^* \mathcal{L}^{-1})$ is a direct summand of $H^i(X_2, p_2^* p_1^* \mathcal{L}^{-1}(bD))$.

$$p_2^* p_1^* \mathcal{L}(bD) = \underbrace{p_2^* p_1^* M}_{\text{big and nef}} + \underbrace{\sum_{i \geq 1} a_i \, p_2^* p_1^* (D_i)}_{\text{snc} \quad 0 \leq a_i < 1}.$$

M big and nef

$$\boxed{\underset{\text{ample}}{M \sim_{\mathbb{Q}} A} + \underset{\text{eff.}}{E}}$$

There exists $f: Y \to X$ proj birational s.t.

$f^* \mathcal{L} = A + E$, where A ample and E snc

$E = \sum a_i E_i \quad 0 \le a_i < 1$

$H \subseteq X$ an ample divisor

$$H^i(X, \mathcal{L}(rH) \otimes R^j f_* \omega_Y) \Longrightarrow H^{i+j}(Y, \omega_Y \otimes f^* \mathcal{L}(rH)).$$

$$f^* \mathcal{L}(rH) = \underbrace{(A + r f^* H)}_{\text{nef + ample = ample.}} + E$$

using the version of KV that we already proved.

$$H^k(Y, f^* \mathcal{L}(rH) \otimes \omega_Y) = 0 \quad \text{for } k > 0.$$

$r \gg 0$

$$H^0(X, \underbrace{\mathcal{L}(rH)}_{\text{very ample}} \otimes R^j f_* \omega_Y) = H^j(Y, \omega_Y \otimes f^* \mathcal{L}(rH)) = 0$$

$\| $
0

$R^j f_* \omega_Y = 0$ for $j > 0$. $\quad r = 0.$

$$H^i(X, \mathcal{L} \otimes f_* \omega_Y) \simeq H^i(Y, f^* \mathcal{L} \otimes \omega_Y) = 0 \quad \square$$

Theorem: (X, Δ) proper klt pair. N $\mathbb{Q}$-Carbier Weil.

$N = M + \Delta$, where M is big and nef. $\mathbb{Q}$-Carbier $\mathbb{Q}$-divisor.

Then $H^i(X, \mathcal{O}_X(-N)) = 0$ for $i < \dim X$.

is

$$\boxed{H^{n-i}(X, K_X + N) = 0 \qquad n-i > 0}$$

klt $\Longrightarrow$ rat sing $\Longrightarrow$ CM.

lc $\Longrightarrow$ DuBois. Hodge Th